

KHALIFA UNIVERSITY OF SCIENCE, TECHNOLOGY AND RESEARCH

MATH 204-LINEAR ALGEBRA- SPRING 2013

Home Work 1- 13 February, 2013

DUE ON: 18 February, 2013

1. Which of the following are linear equations?

(a) $x_1 + 2x_1x_2 + x_3 = 2$ (b) $x_1 = \sqrt{2}x_3 - x_2 + 7$ (c) $x_1 + x_2^{-1} - 3x_3 = 5$

2. Find the solution set for $2x_1 + 4x_2 - 7x_3 = 8$

3. Find the augmented matrix for the following systems of linear equations

(a) $x_1 - 2x_2 = 0$; $3x_1 + 1 + 4x_2 = 0$; $2x_1 - x_2 = 3$

(b) $x_1 + x_3 = 1$; $2x_2 - x_3 + x_5 = 2$; $2x_3 + x_4 = 3$

(c) $x_1 = 1$; $x_2 = 2$

4. Find a system of linear equations corresponding to the augmented matrix

$$\left(\begin{array}{cccc} 1 & 0 & -1 & 2 \\ 2 & 1 & 1 & 3 \end{array} \right)$$

5. Consider the system of equations

$$ax + by = k$$

$$cx + dy = l$$

$$ex + fy = m$$

Discuss the relative positions of the lines $ax+by=k$, $cx+dy=l$, $ex+fy=m$, when:

(a) The system has no solutions

(b) The system has exactly one solution

(c) The system has infinitely many solutions

6. Solve each of the following equations by Gauss-Jordan elimination:

(a) $x_1 + x_2 + 2x_3 = 8$; $-x_1 - 2x_2 + 3x_3 = 1$; $3x_1 - 7x_2 + 4x_3 = 10$

(b) $x_1 - 2x_2 + x_3 = 0$; $2x_2 - 8x_3 = 8$; $-4x_1 + 5x_2 + 9x_3 = -9$

(c) $3x_1 + 2x_2 - x_3 = -15$; $5x_1 + 3x_2 + 2x_3 = 0$; $3x_1 + x_2 + 3x_3 = 11$; $11x_1 + 7x_2 = -30$

7. For which values of "a" will the following system has no solutions? Exactly one Solution? Infinitely many solutions?

$$x + 2y - 3z = 4, 3x - y + 5z = 2, 4x + y + (a^2 - 14)z = a + 2$$

8. Reduce to reduced row-echelon form.

$$\left(\begin{array}{cccccc} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array} \right)$$

Due: 18 Feb 2013.

SolutionsQ-1

- a) Not linear because of Product term ($x_1 x_2$)
 b) Linear
 c) Not linear because variable (x_2) in denominator

Q-2

$$2x_1 + 4x_2 - 7x_3 = 8.$$

of Eqn = 1

Underdetermined & typically we expect

of unknowns = 3

Definitely many number of solution.

$$2x_1 = 8 - 4x_2 + 7x_3$$

$$x_1 = \frac{1}{2} (8 - 4x_2 + 7x_3).$$

Assigning arbitrary values for x_2 & x_3

$$x_2 = s = 0, 1, 2, \dots$$

$$x_3 = t = 0, 1, 2, \dots$$

$$\text{Then } x_1 = \frac{1}{2} (8 - 4s + 7t).$$

$$x_2 = s$$

$$\text{where } s, t = 0, 1, 2, \dots$$

$$x_3 = t$$

when $x_2 = x_3 = 0$ i.e. $s = t = 0$ $\therefore$ Infinitely many solutions.

$$x_1 = 2$$

Q-3

a) $x_1 - 2x_2 = 0$
 $3x_1 + 1 + 4x_2 = 0$
 $2x_1 - x_2 = 3$

$$B = [A|b] = \begin{pmatrix} x_1 & x_2 & b \\ 1 & -2 & 0 \\ 3 & 4 & -1 \\ 2 & -1 & 3 \end{pmatrix}$$

b) $x_1 + x_3 = 1$
 $2x_2 - x_3 + x_5 = 2$
 $2x_3 + x_4 = 3$

$$B = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & b \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 2 & -1 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 & 0 & 3 \end{pmatrix}$$

Max. # of variables = 5

c) $x_1 = 1$
 $x_2 = 2$

$$B = \begin{pmatrix} x_1 & x_2 & b \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

Q-4

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 1 & 3 \end{pmatrix}$$

$$x_1 - x_3 = 2$$

$$2x_1 + x_2 + x_3 = 3$$

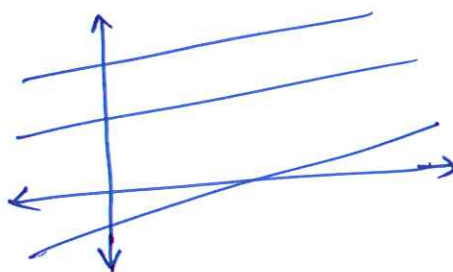
Q-5

$$ax + by = k$$

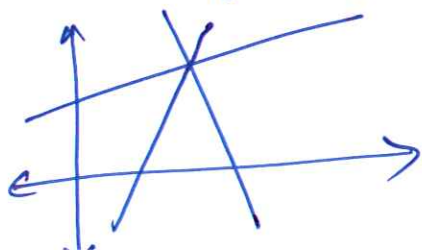
$$cx + dy = l$$

$$ex + fy = m$$

a) No solution, when the three lines do not intersect each other



b) Exactly one solution, when the three lines intersect at exactly one point



c) Infinitely many solutions when the three lines are coincident.

Note: If the rhs = 0; i.e. $k = l = m = 0$ then the system has at least one solution $x = y = 0$ (Trivial solution)

Q-6)

(2)

a) Gauss-Jordan Elimination method requires to convert the Augmented matrix to row reduced Echelon form using elementary row operations.

$$a) \left[\begin{array}{ccc|c} \boxed{1} & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & \boxed{-1} & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array} \right]$$

$$R_2 \rightarrow R_2 + R_1; R_3 \rightarrow R_3 - 3R_1$$

$$\downarrow R_2 \rightarrow (-R_2)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & -10 & -2 & -14 \end{array} \right]$$

$$R_1 \rightarrow R_1 - R_2; R_3 \rightarrow R_3 + 10R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & \boxed{-52} & -104 \end{array} \right]$$

$$R_3 \rightarrow R_3 / (-52)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 7 & 17 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & \boxed{1} & 2 \end{array} \right]$$

$$R_1 \rightarrow R_1 - 7R_3; R_2 \rightarrow R_2 + 5R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$\therefore \begin{aligned} x_1 &= 3 \\ x_2 &= 1 \\ x_3 &= 2 \end{aligned}$$

Unique solution

Q-6 (b)

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array} \right]$$

$$R_3 \rightarrow R_3 + 4R_1$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array} \right] \rightarrow$$

$$R_2 \rightarrow R_2/2$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array} \right]$$

$$R_1 \rightarrow R_1 + 2R_2$$

$$R_3 \rightarrow R_3 + 3R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

$$R_1 \rightarrow R_1 + 7R_3, R_2 \rightarrow R_2 + 4R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array} \right]$$

$$x_1 = 29, x_2 = 16, x_3 = 3$$

Unique solution.

Q-6 (C)

Row
Operation
1:

$$\begin{array}{cccc} 3 & 2 & -1 & -15 \\ 5 & 3 & 2 & 0 \\ 3 & 1 & 3 & 11 \\ 11 & 7 & 0 & -30 \end{array}$$

multiply the 1st row by $\frac{1}{3}$

$$\begin{array}{cccc} 2 & -1 & & \\ 1 & \frac{2}{3} & \frac{-1}{3} & -5 \\ 3 & 3 & & \\ 5 & 3 & 2 & 0 \\ 3 & 1 & 3 & 11 \\ 11 & 7 & 0 & -30 \end{array}$$

Row
Operation
2:

$$\begin{array}{cccc} 2 & -1 & & \\ 1 & \frac{2}{3} & \frac{-1}{3} & -5 \\ 3 & 3 & & \\ 5 & 3 & 2 & 0 \\ 3 & 1 & 3 & 11 \\ 11 & 7 & 0 & -30 \end{array}$$

add -5 times the 1st row to the 2nd row

$$\begin{array}{cccc} 2 & -1 & & \\ 1 & \frac{2}{3} & \frac{-1}{3} & -5 \\ 3 & 3 & & \\ & -1 & 11 & \\ 0 & \frac{2}{3} & \frac{-1}{3} & 25 \\ 3 & 3 & & \\ 3 & 1 & 3 & 11 \\ 11 & 7 & 0 & -30 \end{array}$$

Row
Operation
3:

$$\begin{array}{cccc} 2 & -1 & & \\ 1 & \frac{2}{3} & \frac{-1}{3} & -5 \\ 3 & 3 & & \\ & -1 & 11 & \\ 0 & \frac{2}{3} & \frac{-1}{3} & 25 \\ 3 & 3 & & \\ 3 & 1 & 3 & 11 \\ 11 & 7 & 0 & -30 \end{array}$$

add -3 times the 1st row to the 3rd row

$$\begin{array}{cccc} 2 & -1 & & \\ 1 & \frac{2}{3} & \frac{-1}{3} & -5 \\ 3 & 3 & & \\ & -1 & 11 & \\ 0 & \frac{2}{3} & \frac{-1}{3} & 25 \\ 3 & 3 & & \\ 0 & -1 & 4 & 26 \\ 11 & 7 & 0 & -30 \end{array}$$

Row
Operation
4:

$$\begin{array}{cccc} 2 & -1 & & \\ 1 & \frac{2}{3} & \frac{-1}{3} & -5 \\ 3 & 3 & & \\ & -1 & 11 & \\ 0 & \frac{2}{3} & \frac{-1}{3} & 25 \\ 3 & 3 & & \\ 0 & -1 & 4 & 26 \\ 11 & 7 & 0 & -30 \end{array}$$

add -11 times the 1st row to the 4th row

$$\begin{array}{cccc} 2 & -1 & & \\ 1 & \frac{2}{3} & \frac{-1}{3} & -5 \\ 3 & 3 & & \\ & -1 & 11 & \\ 0 & \frac{2}{3} & \frac{-1}{3} & 25 \\ 3 & 3 & & \\ 0 & -1 & 4 & 26 \\ & -1 & 11 & \\ 0 & \frac{2}{3} & \frac{-1}{3} & 25 \\ 3 & 3 & & \end{array}$$

Row
Operation
5:

$$\begin{array}{cccc} 2 & -1 & & \\ 1 & \frac{1}{3} & \frac{1}{3} & -5 \\ 3 & 3 & & \\ -1 & 11 & & \\ 0 & \frac{1}{3} & \frac{1}{3} & 25 \\ 3 & 3 & & \\ 0 & -1 & 4 & 26 \\ -1 & 11 & & \\ 0 & \frac{1}{3} & \frac{1}{3} & 25 \\ 3 & 3 & & \end{array}$$

multiply the 2nd row by -3

$$\begin{array}{cccc} 2 & -1 & & \\ 1 & \frac{1}{3} & \frac{1}{3} & -5 \\ 3 & 3 & & \\ 0 & 1 & -11 & -75 \\ 0 & -1 & 4 & 26 \\ -1 & 11 & & \\ 0 & \frac{1}{3} & \frac{1}{3} & 25 \\ 3 & 3 & & \end{array}$$

Row
Operation
6:

$$\begin{array}{cccc} 2 & -1 & & \\ 1 & \frac{1}{3} & \frac{1}{3} & -5 \\ 3 & 3 & & \\ 0 & 1 & -11 & -75 \\ 0 & -1 & 4 & 26 \\ -1 & 11 & & \\ 0 & \frac{1}{3} & \frac{1}{3} & 25 \\ 3 & 3 & & \end{array}$$

add 1 times the 2nd row to the 3rd row

$$\begin{array}{cccc} 2 & -1 & & \\ 1 & \frac{1}{3} & \frac{1}{3} & -5 \\ 3 & 3 & & \\ 0 & 1 & -11 & -75 \\ 0 & 0 & -7 & -49 \\ -1 & 11 & & \\ 0 & \frac{1}{3} & \frac{1}{3} & 25 \\ 3 & 3 & & \end{array}$$

Row
Operation
7:

$$\begin{array}{cccc} 2 & -1 & & \\ 1 & \frac{1}{3} & \frac{1}{3} & -5 \\ 3 & 3 & & \\ 0 & 1 & -11 & -75 \\ 0 & 0 & -7 & -49 \\ -1 & 11 & & \\ 0 & \frac{1}{3} & \frac{1}{3} & 25 \\ 3 & 3 & & \end{array}$$

add $\frac{1}{3}$ times the 2nd row to the 4th row

$$\begin{array}{cccc} 2 & -1 & & \\ 1 & \frac{1}{3} & \frac{1}{3} & -5 \\ 3 & 3 & & \\ 0 & 1 & -11 & -75 \\ 0 & 0 & -7 & -49 \\ 0 & 0 & 0 & 0 \\ -1 & 11 & & \\ 0 & \frac{1}{3} & \frac{1}{3} & 25 \\ 3 & 3 & & \end{array}$$

Row
Operation
8:

$$\begin{array}{cccc} 2 & -1 & & \\ 1 & \frac{1}{3} & \frac{1}{3} & -5 \\ 3 & 3 & & \\ 0 & 1 & -11 & -75 \\ 0 & 0 & -7 & -49 \\ 0 & 0 & 0 & 0 \end{array}$$

multiply the 3rd row by $-\frac{1}{7}$

$$\begin{array}{cccc} 2 & -1 & & \\ 1 & \frac{1}{3} & \frac{1}{3} & -5 \\ 3 & 3 & & \\ 0 & 1 & -11 & -75 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{array}$$

Row
Operation

$$\begin{array}{cccc} 1 & 2 & -1 & -5 \end{array}$$

add 11 times the 3rd row to the 2nd row

$$\begin{array}{cccc} 1 & 2 & -1 & -5 \end{array}$$

9:

$$\begin{array}{cccc|c} & \overline{} & \overline{} & & \\ 3 & 3 & & & \\ 0 & 1 & -11 & -75 & \\ 0 & 0 & 1 & 7 & \\ 0 & 0 & 0 & 0 & \end{array}$$

$$\begin{array}{cccc|c} & \overline{} & \overline{} & & \\ 3 & 3 & & & \\ 0 & 1 & 0 & 2 & \\ 0 & 0 & 1 & 7 & \\ 0 & 0 & 0 & 0 & \end{array}$$

Row
Operation
10:

$$\begin{array}{cccc|c} 2 & -1 & & & \\ 1 & \overline{} & \overline{} & -5 & \\ 3 & 3 & & & \\ 0 & 1 & 0 & 2 & \\ 0 & 0 & 1 & 7 & \\ 0 & 0 & 0 & 0 & \end{array}$$

add $\frac{1}{3}$ times the 3rd row to the 1st row

$$\begin{array}{cccc|c} 2 & -8 & & & \\ 1 & \overline{} & 0 & \overline{} & \\ 3 & 3 & & & \\ 0 & 1 & 0 & 2 & \\ 0 & 0 & 1 & 7 & \\ 0 & 0 & 0 & 0 & \end{array}$$

Row
Operation
11:

$$\begin{array}{cccc|c} 2 & -8 & & & \\ 1 & \overline{} & 0 & \overline{} & \\ 3 & 3 & & & \\ 0 & 1 & 0 & 2 & \\ 0 & 0 & 1 & 7 & \\ 0 & 0 & 0 & 0 & \end{array}$$

add $-\frac{2}{3}$ times the 2nd row to the 1st row

$$\begin{array}{cccc|c} 1 & 0 & 0 & -4 & \\ 0 & 1 & 0 & 2 & \\ 0 & 0 & 1 & 7 & \\ 0 & 0 & 0 & 0 & \end{array}$$

Q-7

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-4 & a+2 \end{array} \right)$$

(5)

Reducing the matrix to row Echelon form

$$R_2 \rightarrow R_2 - 3R_1, \quad R_3 \rightarrow R_3 - 4R_1$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & (a^2-2) & (a-4) \end{array} \right)$$

$$R_3 \rightarrow R_3 - R_2$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & (a^2-16) & (a-4) \end{array} \right)$$

$$(a^2-16)x_3 = (a-4) \Rightarrow x_3 = \frac{(a-4)}{(a^2-16)} = \frac{1}{(a+4)}$$

 $\therefore$ very backward substitution -

$$x_2 = \frac{10a+54}{7(a+4)}, \quad x_1 = \frac{8a+15}{7(a+4)}$$

when $a = -4$

$$0 \cdot x_3 = -8; \quad \underline{\text{No solution}}$$

when $a = +4$

$$x_3 = 1/8, \quad x_2 = 47/28, \quad x_1 = 47/56 \rightarrow \underline{\text{Unique solution}}$$

when $a \neq \pm 4$

$$\text{Then } x_3 = t = \frac{1}{a+4}; \quad x_2 = \frac{10+14t}{7}$$

$$x_1 = 8 - \frac{7t}{7}$$

Infinity
Many solutions

Q-8

6

Row
Operation
1:

$$\begin{array}{cccccc} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}$$

interchange the 1st row and the
2nd row

$$\begin{array}{cccccc} 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}$$

Row
Operation
2:

$$\begin{array}{cccccc} 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}$$

multiply the 1st row by $1/3$

$$\begin{array}{cccccc} -7 & 8 & -5 & 8 & & \\ 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 3 \\ 3 & 3 & 3 & 3 & & \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}$$

Row
Operation
3:

$$\begin{array}{cccccc} -7 & 8 & -5 & 8 & & \\ 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 3 \\ 3 & 3 & 3 & 3 & & \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}$$

add -3 times the 1st row to the 3rd row

$$\begin{array}{cccccc} -7 & 8 & -5 & 8 & & \\ 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 3 \\ 3 & 3 & 3 & 3 & & \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 0 & -2 & 4 & -4 & -2 & 6 \end{array}$$

Row
Operation
4:

$$\begin{array}{cccccc} -7 & 8 & -5 & 8 & & \\ 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 3 \\ 3 & 3 & 3 & 3 & & \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 0 & -2 & 4 & -4 & -2 & 6 \end{array}$$

multiply the 2nd row by $1/3$

$$\begin{array}{cccccc} -7 & 8 & -5 & 8 & & \\ 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 3 \\ 3 & 3 & 3 & 3 & & \\ & & & & 4 & -5 \\ 0 & 1 & -2 & 2 & \frac{1}{3} & \frac{1}{3} \\ & & & & 3 & 3 \\ 0 & -2 & 4 & -4 & -2 & 6 \end{array}$$

Row
Operation
5:

$$\begin{array}{cccccc} -7 & 8 & -5 & 8 & & \\ 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 3 \\ 3 & 3 & 3 & 3 & & \\ & & & & 4 & -5 \\ 0 & 1 & -2 & 2 & \frac{1}{3} & \frac{1}{3} \\ & & & & 3 & 3 \\ 0 & -2 & 4 & -4 & -2 & 6 \end{array}$$

add 2 times the 2nd row to the 3rd row

$$\begin{array}{cccccc} -7 & 8 & -5 & 8 & & \\ 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 3 \\ 3 & 3 & 3 & 3 & & \\ & & & & 4 & -5 \\ 0 & 1 & -2 & 2 & \frac{1}{3} & \frac{1}{3} \\ & & & & 3 & 3 \\ & & & & 2 & 8 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \end{array}$$

Row
Operation
6:

$$\begin{array}{cccc|cccc} & -7 & 8 & -5 & 8 & & & \\ 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 3 & & \\ & 3 & 3 & 3 & 3 & & & \\ & & & & 4 & -5 & & \\ 0 & 1 & -2 & 2 & \frac{1}{3} & \frac{1}{3} & & \\ & & & & 2 & 8 & & \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & & \end{array}$$

multiply the 3rd row by $\frac{3}{2}$

$$\begin{array}{cccc|cccc} & -7 & 8 & -5 & 8 & & & \\ 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 3 & & \\ & 3 & 3 & 3 & 3 & & & \\ & & & & 4 & -5 & & \\ 0 & 1 & -2 & 2 & \frac{1}{3} & \frac{1}{3} & & \\ & & & & 3 & 3 & & \\ 0 & 0 & 0 & 0 & 1 & 4 & & \end{array}$$

Row
Operation
7:

$$\begin{array}{cccc|cccc} & -7 & 8 & \frac{1}{3} & 8 & & & \\ 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 3 & & \\ & 3 & 3 & \frac{1}{3} & \frac{1}{3} & & & \\ & & & & 4 & \frac{1}{3} & & \\ 0 & 1 & \frac{1}{2} & 2 & \frac{1}{3} & \frac{1}{3} & & \\ & & & & 3 & 3 & & \\ 0 & 0 & 0 & 0 & 1 & 4 & & \end{array}$$

add $-\frac{4}{3}$ times the 3rd row to the 2nd row

$$\begin{array}{cccc|cccc} & \frac{1}{3} & 8 & -5 & 8 & & & \\ 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 3 & & \\ & \frac{1}{3} & 3 & 3 & 3 & & & \\ & & & & 4 & \frac{1}{3} & & \\ 0 & 1 & -2 & 2 & 0 & -7 & & \\ 0 & 0 & 0 & 0 & 1 & 4 & & \end{array}$$

Row
Operation
8:

$$\begin{array}{cccc|cccc} & \frac{1}{3} & 8 & \frac{1}{3} & 8 & & & \\ 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 3 & & \\ & \frac{1}{3} & 3 & \frac{1}{3} & \frac{1}{3} & & & \\ & & & & 4 & \frac{1}{3} & & \\ 0 & 1 & \frac{1}{2} & 2 & 0 & \frac{1}{3} & & \\ & & & & 3 & 3 & & \\ 0 & 0 & 0 & 0 & 1 & 4 & & \end{array}$$

add $-\frac{8}{3}$ times the 3rd row to the 1st row

$$\begin{array}{cccc|cccc} & \frac{1}{3} & 8 & \frac{1}{3} & \frac{1}{3} & & & \\ 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 3 & & \\ & \frac{1}{3} & 3 & \frac{1}{3} & \frac{1}{3} & & & \\ & & & & 4 & \frac{1}{3} & & \\ 0 & 1 & \frac{1}{2} & 2 & 0 & -7 & & \\ 0 & 0 & 0 & 0 & 1 & 4 & & \end{array}$$

Row
Operation
9:

$$\begin{array}{cccc|cccc} & -7 & 8 & -5 & -23 & & & \\ 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & & \\ & 3 & 3 & 3 & & 3 & & \\ 0 & 1 & -2 & 2 & 0 & -7 & & \\ 0 & 0 & 0 & 0 & 1 & 4 & & \end{array}$$

add $\frac{7}{3}$ times the 2nd row to the 1st row

$$\begin{array}{cccc|cccc} & 1 & 0 & -2 & 3 & 0 & -24 & \\ 1 & 0 & -2 & 3 & 0 & -24 & & \\ 0 & 1 & -2 & 2 & 0 & -7 & & \\ 0 & 0 & 0 & 0 & 1 & 4 & & \end{array}$$