KHALIFA UNIVERSITY OF SCIENCE, TECHNOLOGY AND RESEARCH
MATH 204-LINEAR ALGEBRA- SPRING 2013
Home Work with Solutions 4- 24 March, 2013
Submissions Due on 1 April, 2013

1. Find a initial point P with terminal point $Q(3,0,-5)$ such that
   a. $\vec{PQ}$ is equal to $\mathbf{v}=(4,-2,-1)$
   b. $\vec{PQ}$ is equal to $\mathbf{v}=(4,-2,-1)$ with oppositely directed.

Sol)
a) Let $\mathbf{P}=(x,y,z)$ be the initial point of the desired vector.

$\vec{PQ}=(3-x,\ 0-y,\ -5-z)=(4,\ -2,\ -1)$. Thus, $\mathbf{P}=(-1,2,-4)$.

b) Let $\mathbf{P}=(x,y,z)$ be the initial point of the desired vector.

$\vec{PQ}=(3-x,\ 0-y,\ -5-z)=-(4,\ -2,\ -1)$. Thus, $\mathbf{P}=(7,-2,-6)$.

2. Let $\mathbf{u}=(-3,1,2,4,4)$, $\mathbf{v}=(4,0,-8,1,2)$, and $\mathbf{w}=(6,-1,-4,3,-5)$. Find the components of the vector $\mathbf{x}$ that satisfies the equation $2\mathbf{u}+\mathbf{v}+\mathbf{x}=6\mathbf{x}+\mathbf{w}$.

Sol) Let $\mathbf{x}=(x_1,x_2,x_3,x_4,x_5)$. Then

$2(-3,1,2,4,4)+(4,0,-8,1,2)+(x_1,x_2,x_3,x_4,x_5)=6(x_1,x_2,x_3,x_4,x_5)+(6,-1,-4,3,-5)$.

If we equate the components of these two vectors, we obtain

$$-2+x_1=6x_1+6$$

$$2+x_2=6x_2-1$$

$$-4+x_3=6x_3-4$$

$$9+x_4=6x_4+3$$

$$10+x_5=6x_5-5$$

Therefore, $\mathbf{x}=(x_1,x_2,x_3,x_4,x_5)=(-\frac{8}{5},\frac{3}{5},0,\frac{6}{5},3)$.

3. Show that there do not exist scalars c_1, c_2, c_3 such that

$$c_1(-2,9,6)+c_2(-3,2,1)+c_3(1,7,5)=(0,5,4)$$

Sol) Suppose there are scalars c_1, c_2, and c_3 which satisfy the given equation. If we equate components on both sides, we obtain the following system of equations:

$$\begin{aligned} -2c_1 - 3c_2 + c_3 &= 0 \\ 9c_1 + 2c_2 + 7c_3 &= 5 \\ 6c_1 + c_2 + 5c_3 &= 4 \end{aligned}$$

The augmented matrix of this system of equations can be reduced to

$$\begin{bmatrix} 2 & 3 & -1 & 0 \\ 9 & 2 & 7 & 5 \\ 6 & 1 & 5 & 4 \end{bmatrix} \xrightarrow{\substack{R_2=-2R_2+3R_3 \\ R_3=-3R_1+R_3}} \begin{bmatrix} 2 & 3 & -1 & 0 \\ 0 & -1 & 1 & 2 \\ 0 & -8 & 8 & 4 \end{bmatrix} \xrightarrow{\substack{R_2=-R_2 \\ R_3=-8R_2+R_3}} \begin{bmatrix} 2 & 3 & -1 & 0 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & -12 \end{bmatrix}$$

The third row of the above matrix implies that $0c_1 + 0c_2 + 0c_3 = -12$. Clearly, there do not exist scalars c_1, c_2, and c_3 which satisfy the above equation, and hence the system is inconsistent.

4. Let $\mathbf{v}=(0,2,-6,3)$. Find all scalars k such that $\|k\mathbf{v}\|=14$.

Sol) Since $k\mathbf{v}=(0,2k,-6k,3k)$, then $\|k\mathbf{v}\|=\sqrt{4k^2+36k^2+9k^2}=\sqrt{49k^2}=|k|7$.

If $\|k\mathbf{v}\|=14$, it follows that $7|k|=14$, so k=2 and -2.

5. Find a unit vector that has the same direction as the given vector.
   a. $\mathbf{u}$=(3,-4)
   b. $\mathbf{v}$= (3,6,-2)

Sol) a) Since $\|\mathbf{u}\|=\sqrt{9+16}=5$, then $\frac{1}{\|\mathbf{u}\|}\mathbf{u}=\left(\frac{3}{5},-\frac{4}{5}\right)$.

b) Since $\|\mathbf{v}\|=\sqrt{9+36+4}=7$, then $\frac{1}{\|\mathbf{v}\|}\mathbf{v}=\left(\frac{3}{7},\frac{6}{7},\frac{-2}{7}\right)$

6. Let $\mathbf{p}_0 = (x_0, y_0, z_0)$ and $\mathbf{p} = (x, y, z)$. Describe the set of all points (x,y,z) for which $\|\mathbf{p} - \mathbf{p}_0\| = 1$.

Sol) Note that $\|\mathbf{p} - \mathbf{p}_0\| = 1$ if and only if $\|\mathbf{p} - \mathbf{p}_0\|^2 = 1$. Thus

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = 1$$

The points (x, y, z) which satisfy these equations are just the points on the sphere of radius 1 with center (x_0, y_0, z_0); that is, they are all the points whose distance from (x_0, y_0, z_0) is 1.

7. Let $\mathbf{u} = (2, -1, 3)$ and $\mathbf{v} = (4, -1, 2)$. Find the vector component of **u** along **v** and the vector component of **u** orthogonal to **v.**

   Sol) The vector component of **u** along **v** is

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\bullet\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{2*4 + (-1)*(-1) + 3*2}{4^2 + (-1)^2 + 2^2}\mathbf{v} = \frac{15}{21}(4, -1, 2) = \left(\frac{20}{7}, -\frac{5}{7}, \frac{10}{7}\right).$$

The vector component of **u** orthogonal to **v**

$$\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u} = \mathbf{u} - \frac{\mathbf{u}\bullet\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = (2, -1, 3) - \left(\frac{20}{7}, -\frac{5}{7}, \frac{10}{7}\right) = \left(-\frac{6}{7}, -\frac{2}{7}, \frac{11}{7}\right).$$

8. Determine whether the vectors form an orthogonal set.

   a) $\mathbf{v}_1 = (1, 2),\ \mathbf{v}_2 = (-2, 1)$

   b) $\mathbf{v}_1 = (2, 4),\ \mathbf{v}_2 = (-1, 2)$

   c) $\mathbf{v}_1 = (1, 3, -1),\ \mathbf{v}_2 = (-2, 2, 4),\ \mathbf{v}_3 = (14, -2, 8)$

   d) $\mathbf{v}_1 = (5, -1, 2),\ \mathbf{v}_2 = (-1, 2, 3),\ \mathbf{v}_3 = (4, -1, 2)$

   Sol)

   a. $\mathbf{v}_1\bullet\mathbf{v}_2 = (1*-2) + (1*2) = 0$, Orthogonal set

   b. $\mathbf{v}_1\bullet\mathbf{v}_2 = (2*-1) + (4*2) = 8$, Not orthogonal set

c. $\mathbf{v}_1 \bullet \mathbf{v}_2 = (1*-2)+(3*2)+(1*4)=0$, $\mathbf{v}_1 \bullet \mathbf{v}_3 = (1*14)+(3*-2)+(-1*8)=0$, $\mathbf{v}_2 \bullet \mathbf{v}_3 = (-2*14)+(2*-2)+(4*8)=0$, Orthogonal set

d. $\mathbf{v}_1 \bullet \mathbf{v}_2 = (5*-1)+(-1*2)+(2*3)=-1$, $\mathbf{v}_1 \bullet \mathbf{v}_3 = (5*4)+(-1*-1)+(2*2)=25$, $\mathbf{v}_2 \bullet \mathbf{v}_3 = (-1*4)+(2*-1)+(3*2)=0$, Not orthogonal set

9. Show that if **v** is orthogonal to both $\mathbf{w}_1$ and $\mathbf{w}_2$, then **v** is orthogonal to $k_1\mathbf{w}_1 + k_2\mathbf{w}_2$ for all scalars k_1 and k_2.

Sol) Note that $\mathbf{v} \bullet (k_1\mathbf{w}_1 + k_2\mathbf{w}_2) = k_1(\mathbf{v} \bullet \mathbf{w}_1) + k_2(\mathbf{v} \bullet \mathbf{w}_2) = 0$ because, by hypothesis, $\mathbf{v} \bullet \mathbf{w}_1 = \mathbf{v} \bullet \mathbf{w}_2 = 0$. Therefore **v** is orthogonal to $k_1\mathbf{w}_1 + k_2\mathbf{w}_2$ for any scalars k_1 and k_2.

10. Let $\mathbf{u} = (3,2,-1)$, $\mathbf{v} = (0,2,-3)$, and $\mathbf{w} = (2,6,7)$ .Compute the indicated vectors

a) $\mathbf{v} \times \mathbf{w}$

b) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$

c) $\mathbf{u} \times (\mathbf{v} - 2\mathbf{w})$

Sol) a. $\mathbf{v} \times \mathbf{w} = \left(\begin{vmatrix} 2 & -3 \\ 6 & 7 \end{vmatrix}, -\begin{vmatrix} 0 & -3 \\ 2 & 7 \end{vmatrix}, \begin{vmatrix} 0 & 2 \\ 2 & 6 \end{vmatrix} \right) = (32,-6,-4)$

b. Since $(\mathbf{u} \times \mathbf{v}) = \left(\begin{vmatrix} 2 & -1 \\ 2 & -3 \end{vmatrix}, -\begin{vmatrix} 3 & -1 \\ 0 & -3 \end{vmatrix}, \begin{vmatrix} 3 & 2 \\ 0 & 2 \end{vmatrix} \right) = (-4,9,6)$. We have

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \left(\begin{vmatrix} 9 & 6 \\ 6 & 7 \end{vmatrix}, -\begin{vmatrix} -4 & 6 \\ 2 & 7 \end{vmatrix}, \begin{vmatrix} -4 & 9 \\ 2 & 6 \end{vmatrix} \right) = (27,40,-42).$$

c. Since $(\mathbf{u} - 2\mathbf{w}) = (0,2,-3)-(4,12,14) = (-4,-10,-17)$. We have

$$\mathbf{u} \times (\mathbf{v} - 2\mathbf{w}) = \left(\begin{vmatrix} 2 & -1 \\ -10 & -17 \end{vmatrix}, -\begin{vmatrix} 3 & -1 \\ -4 & -17 \end{vmatrix}, \begin{vmatrix} 3 & 2 \\ -4 & -10 \end{vmatrix} \right) = (-44,-10,-17).$$

11. Determine whether the given set of vectors is linearly independent or linearly dependent.

a. $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

b. $u_1 = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$, $u_2 = \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix}$, and $u_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Sol) a. $\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 3 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -1 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 + 2R_2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 0 \end{array}\right] \xrightarrow{R_2 \to -R_2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$, Only trivial solution, thus $\{v_1, v_2\}$ is linearly independent.

b.

$$\left(\begin{array}{ccc|c} 2 & 4 & 1 & 0 \\ 1 & 4 & 1 & 0 \\ -3 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 4 & 1 & 0 \\ 2 & 4 & 1 & 0 \\ -3 & 0 & 0 & 0 \end{array}\right) \xrightarrow[R_3 \to R_3 + 3R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 4 & 1 & 0 \\ 0 & -4 & -1 & 0 \\ 0 & 12 & 3 & 0 \end{array}\right) \xrightarrow{R_2 \to -\frac{1}{4}R_2} \left(\begin{array}{ccc|c} 1 & 4 & 1 & 0 \\ 0 & 1 & 1/4 & 0 \\ 0 & 12 & 3 & 0 \end{array}\right) \xrightarrow[R_3 \to R_3 - 12R_2]{R_1 \to R_1 - 4R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus, thus $\{u_1, u_2, u_3\}$ is linearly dependent since there are solutions in addition to the trivial solution. For example, $x_1 = 0, x_2 = -\frac{1}{4}, x_3 = 1$.

12. Determine which of the following are subspaces of R^3.

a. All vectors of the form (a, 0,0)
b. All vectors of the form (a,1,1)
c. All vectors of the form (a,b,0)

Sol).

a) The set is closed under vector addition because (a, 0, 0) + (b, 0, 0) = (a + b, 0, 0).
It is closed under scalar multiplication because *k*(a, 0, 0) = (*k*a, 0, 0). Therefore, it is a subspace of R3.

b) This set is not closed under either vector addition or scalar multiplication.
For example, (a, 1, 1) + (b, 1, 1) = (a + b, 2, 2) and (a + b, 2, 2) does not belong to the set. Thus it is not a subspace.

c) This set is closed under vector addition because (a1, b1, 0) + (a2, b2, 0) = (a1 + a2, b1 + b2, 0). It is also closed under scalar multiplication because *k*(a, b, 0) = (*k*a, *k*b, 0). Therefore, it is a subspace of R3.

13. Let **A** and **B** be 4x5 matrices and let **C**, **D** and **E** be 5x2, 4x2 and 5x4 matrices, respectively. Determine which of the following matrix expressions are defined. For those that are defined, give the size of the resulting matrix.

a. **BA**, **AC**+**D**, **AE**+**B**

b. **AB**+**B**, **E**(**A**+**B**), **E**(**AC**)

Sol)

a) $\mathbf{B}_{4x5}\mathbf{A}_{4x5}$: Because the number of column in B is not equal to the number of row in A, this matrix is not defined.

$\mathbf{A}_{4x5}\mathbf{C}_{5x2}+\mathbf{D}_{4x2}$: 4x2 matrix is defined.

$\mathbf{A}_{4x5}\mathbf{E}_{5x4}+\mathbf{B}_{4x5}$: The matrix is not defined.

b) $\mathbf{A}_{4x5}\mathbf{B}_{4x5}+\mathbf{B}_{4x5}$: The matrix is not defined.

$\mathbf{E}_{5x4}(\mathbf{A}_{4x5}+\mathbf{B}_{4x5})$: 5x5 matrix is defined.

$\mathbf{E}_{5x4}(\mathbf{A}_{4x5}\mathbf{C}_{5x2})$: 5x2 matrix is defined.

14. Show that if **AB** and **BA** are both defined, then **AB** and **BA** are square matrices.

Sol) Let **A** and **B** define *mxn* and *pxq* matrices, respectively. If $\mathbf{A}_{mxn}\mathbf{B}_{pxq}$ is defined, it implies that *n*=*p*. Similarly, if $\mathbf{B}_{pxq}\mathbf{A}_{mxn}$ is defined, it implies that *q*=*m*. Thus, $\mathbf{A}_{mxn}\mathbf{B}_{pxq}=\mathbf{C}_{mxq}$, which is square matrix (since *q*=*m*). As same way, $\mathbf{B}_{pxq}\mathbf{A}_{mxn}=\mathbf{D}_{pxn}$, is square matrix (since *n*=*p*).

15. Solve the following equation $\begin{pmatrix} a-b & b+c \\ 3d+c & 2a-4d \end{pmatrix}-\begin{pmatrix} 8 & 1 \\ 7 & 6 \end{pmatrix}=\mathbf{0}$.

Sol) $\begin{pmatrix} a-b & b+c \\ 3d+c & 2a-4d \end{pmatrix}-\begin{pmatrix} 8 & 1 \\ 7 & 6 \end{pmatrix}=\begin{pmatrix} a-b-8 & b+c-1 \\ 3d+c-7 & 2a-4d-6 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Thus, $a-b-8=0$, $b+c-1=0$, $3d+c-7=0$, and $2a-4d-6=0$

Therefore, a=5, b=-3, c=4, and d=1.

16. Express the following system of equations

$$\begin{aligned} 2x_1+4x_2-2x_3-2&=0 \\ 3x_1+5x_2-5x_3&=1 \\ 2x_2+x_3&=-2 \end{aligned}$$

in the matrix expression of the form $\mathbf{Ax}=\mathbf{b}$, where $\mathbf{x}=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Write down the matrices **A** and **b**.

Sol) The matrix expression of the form $\mathbf{Ax}=\mathbf{b}$ is as follows;

$$\begin{pmatrix} 2 & 4 & -2 \\ 3 & 5 & -5 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}.$$

Therefore, $\mathbf{A}=\begin{pmatrix} 2 & 4 & -2 \\ 3 & 5 & -5 \\ 0 & 2 & 1 \end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$.

17. Prove that $(\mathbf{AB})^T=\mathbf{B}^T\mathbf{A}^T$.

Sol) The *ij*th entry of $\mathbf{B}^T\mathbf{A}^T$ is the scalar product of the *i*th row of $\mathbf{B}^T$ with the *j*th column of $\mathbf{A}^T$. In particular, the *i*th row of $\mathbf{B}^T$ is $[b_{1i}, b_{2i}, \ldots, b_{ni}]$ (the *i*th column of **B**), whereas the *j*th column of $\mathbf{A}^T$ is $\begin{pmatrix} a_{j1} \\ a_{j2} \\ \vdots \\ a_{jn} \end{pmatrix}$ (the *j*th row of **A**). Therefore, the *ij*th entry of $\mathbf{B}^T\mathbf{A}^T$ is given by

$b_{1i}a_{j1}+b_{2i}a_{j2}+\cdots+b_{ni}a_{jn}=\sum_{k=1}^{n} b_{ki}a_{jk}$. Finally, since $\sum_{k=1}^{n} b_{ki}a_{jk}=\sum_{k=1}^{n} a_{jk}b_{ki}$, the *ij*th entries of $(\mathbf{AB})^T$ and $\mathbf{B}^T\mathbf{A}^T$ agree, and the matrices are equal.

18. Let **A** and **B** 2x2 matrices such that $\mathbf{A}^2=\mathbf{AB}$ and $\mathbf{A}\neq\mathbf{0}$. A student proof that if $\mathbf{A}^2=\mathbf{AB}$ then $\mathbf{A}=\mathbf{B}$. The proof is given below: Since $\mathbf{A}^2=\mathbf{AB}$, $\mathbf{A}^2-\mathbf{AB}=\mathbf{0}$ which implies $\mathbf{A}(\mathbf{A}-\mathbf{B})=\mathbf{0}$. Since $\mathbf{A}\neq\mathbf{0}$, it follows that $(\mathbf{A}-\mathbf{B})=\mathbf{0}$. Therefore, $\mathbf{A}=\mathbf{B}$. Is this proof correct?

Sol) No. When $\mathbf{A}(\mathbf{A}-\mathbf{B})=\mathbf{0}$ with $\mathbf{A}\neq\mathbf{0}$, it does not always imply that $(\mathbf{A}-\mathbf{B})=\mathbf{0}$. For example, $\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$.

19. Let $\mathbf{a}=\begin{pmatrix}1\\ 2\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}3\\ 4\end{pmatrix}$, Find $\mathbf{x}=\begin{pmatrix}x_1\\ x_2\end{pmatrix}$ such that $\mathbf{x}^T(\mathbf{a}+\mathbf{b})=12$ and $\mathbf{x}^T\mathbf{a}=2$.

Sol) $\mathbf{x}^T(\mathbf{a}+\mathbf{b})=(x_1\ x_2)\left(\begin{pmatrix}1\\ 2\end{pmatrix}+\begin{pmatrix}3\\ 4\end{pmatrix}\right)=12$, $\mathbf{x}^T\mathbf{a}=(x_1\ x_2)\begin{pmatrix}1\\ 2\end{pmatrix}=2$

Thus, $4x_1+6x_2=12$ and $x_1+2x_2=2$. Therefore, $x_1=6$, $x_2=-2$

20. Let $\mathbf{A}=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$ and $\mathbf{x}=\begin{pmatrix}x_1\\ x_2\end{pmatrix}$. Show that $\mathbf{x}^T\mathbf{A}\mathbf{x}\geq 0$.

Sol) $\mathbf{x}^T\mathbf{A}\mathbf{x}=[x_1\ \ x_2]\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=(x_1+x_2\ \ x_1+x_2)\begin{pmatrix}x_1\\ x_2\end{pmatrix}=[x_1(x_1+x_2)+x_2(x_1+x_2)]$

Therefore, $[x_1(x_1+x_2)+x_2(x_1+x_2)]=x_1^2+2x_1x_2+x_2^2=(x_1+x_2)^2\geq 0$.

21. Let **A** be an invertible matrix whose inverse is $\begin{pmatrix}3 & 4\\ 5 & 6\end{pmatrix}$. Find the matrix A.

Sol) Let **A** define $\mathbf{A}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$. Since **A** is an invertible matrix, $\mathbf{A}\mathbf{A}^{-1}=\mathbf{I}$. Thus,

$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}3 & 4\\ 5 & 6\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. $\begin{pmatrix}3a+5b & 4a+6b\\ 3c+5d & 4c+6d\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

Therefore, a=-3, b=2, c=5/2, and d=-3/2.

22. Find the inverse of $\begin{pmatrix}\cos\theta & \sin\theta\\ \text{-}\sin\theta & \cos\theta\end{pmatrix}$.

Sol) Let $\mathbf{A}=\begin{pmatrix}\cos\theta & \sin\theta\\ \text{-}\sin\theta & \cos\theta\end{pmatrix}$. $\mathbf{A}^{-1}=\dfrac{1}{\cos^2\theta+\sin^2\theta}\begin{pmatrix}\cos\theta & \text{-}\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}=\begin{pmatrix}\cos\theta & \text{-}\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$

23. Assume that **A** is a square matrix and which satisfies $\mathbf{A}^2-3\mathbf{A}+\mathbf{I}=\mathbf{0}$. Show that $\mathbf{A}^{-1}=3\mathbf{I}-\mathbf{A}$.

Sol) $\mathbf{A}^2-3\mathbf{A}+\mathbf{I}=\mathbf{0}\rightarrow\mathbf{A}^{-1}\mathbf{A}\mathbf{A}-3\mathbf{A}^{-1}\mathbf{A}+\mathbf{A}^{-1}\mathbf{I}=\mathbf{A}^{-1}\mathbf{0}$ Multiply $\mathbf{A}^{-1}$ to each term.

$\mathbf{I}\mathbf{A}-3\mathbf{I}+\mathbf{A}^{-1}=\mathbf{0}$. Therefore, $\mathbf{A}^{-1}=3\mathbf{I}-\mathbf{A}$.

24. Prove that if **A** is invertible and $\mathbf{A}\mathbf{B}=\mathbf{0}$, then $\mathbf{B}=\mathbf{0}$.

Sol) Since **A** is invertible, it is possible that $\mathbf{A}^{-1}\mathbf{A}\mathbf{B}=\mathbf{A}^{-1}\mathbf{0}$ => $\mathbf{I}\mathbf{B}=\mathbf{0}$ => $\mathbf{B}=\mathbf{0}$.

25. Let **A** is square matrix and assume $\mathbf{A}^3=3\mathbf{A}^2-3\mathbf{A}+\mathbf{I}$. Express $\mathbf{A}^4$ in terms of $\mathbf{A}^2$, **A**, and **I**.

Sol) $\mathbf{A}^4=\mathbf{A}^3\mathbf{A}=(3\mathbf{A}^2-3\mathbf{A}+\mathbf{I})\mathbf{A}=3\mathbf{A}^3-3\mathbf{A}^2+\mathbf{A}=3(3\mathbf{A}^2-3\mathbf{A}+\mathbf{I})-3\mathbf{A}^2+\mathbf{A}$

$=3(3\mathbf{A}^2-3\mathbf{A}+\mathbf{I})-3\mathbf{A}^2+\mathbf{A}=6\mathbf{A}^2-8\mathbf{A}+3\mathbf{I}$.

26. Show that the given matrix is not invertible. $\mathbf{A}=\begin{pmatrix}1 & 6 & 4\\ 2 & 4 & -1\\ -1 & 2 & 5\end{pmatrix}$

Sol) Using row operations to find $\mathbf{A}^{-1}$,

$$\left(\begin{array}{ccc|ccc}1 & 6 & 4 & 1 & 0 & 0\\ 2 & 4 & -1 & 0 & 1 & 0\\ -1 & 2 & 5 & 0 & 0 & 1\end{array}\right)$$

$$\left(\begin{array}{ccc|ccc}1 & 6 & 4 & 1 & 0 & 0\\ 2 & -8 & -9 & -2 & 1 & 0\\ 0 & 8 & 9 & 1 & 0 & 1\end{array}\right)$$ Added -2 times the first row to the second and added the first row to the third.

$$\left(\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right)$$: Added the second row to the third.

Since we have obtained a row of zeros on the left side, **A** is not invertible.

27. Evaluate the determinants (a) $\det(\mathbf{A}) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix}$, (b) $\det(\mathbf{B}) = \begin{vmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{vmatrix}$.

Sol). (a) $\det(\mathbf{A}) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3\begin{vmatrix} -4 & 3 \\ 4 & 2 \end{vmatrix} - 1\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix} = 3(-4) - (1)(-11) + 0 = -1$

(b) To find det(B) it will be easiest to use cofactor expansion along the second column since it has the most zeros:

$$\det(\mathbf{B}) = \begin{vmatrix} 1 & 0 & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix}$$

For the 3x3 determinant, it will be easiest to use cofactor expansion along its second column since it has the most zeros:

$$\det(\mathbf{B}) = 1 \cdot -2\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = -2(1+2) = -6.$$

28. For which value(s) of k does the matrix $\begin{pmatrix} k-3 & -2 \\ -2 & k-2 \end{pmatrix}$ fail to be invertible?

Sol) Since the given matrix fail to be invertible, it implies that its determinant is zero.

$\det(\mathbf{A}) = (k-3)(k-2) - 4 = 0$, $k^2 - 5k + 2 = 0$. Therefore, $k = \dfrac{5 \pm \sqrt{25-8}}{2} = \dfrac{5 \pm \sqrt{17}}{2}$.

29. Assume that $\mathbf{A}$ is 4x4 matrix and det($\mathbf{A}$)=4. Find

(a) $\det(3\mathbf{A})$ (b) $\det(2\mathbf{A}^{-1})$ (c) $\det((2\mathbf{A})^{-1})$.

Sol)

a) By basic properties of determinants ($\det(k\text{A}) = k^n \det(\text{A})$), $\det(3\mathbf{A}) = 3^4 \det(\mathbf{A}) = 324$.

b) By basic properties of determinants ($\det(\text{A}^{-1}) = \dfrac{1}{\det(\text{A})}$),

$$\det(2\mathbf{A}^{-1}) = 2^4 \det(\mathbf{A}^{-1}) = 16 * \frac{1}{\det(\text{A})} = 4.$$

c) $\det((2\mathbf{A})^{-1}) = \dfrac{1}{\det((2\mathbf{A}))} = \dfrac{1}{2^4 * \det(\mathbf{A})} = \dfrac{1}{64}$.